## Modified Boltzmann Transport Equation

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Recently several works have appeared in the literature in which authors try to describe Freeze Out (FO) in energetic heavy ion collisions based on the Boltzmann Transport Equation (BTE). The aim of this work is to point out the limitations of the BTE, when applied for the modeling of FO or other very fast process, and to propose the way how the BTE approach can be generalized for such a processes.

The Freeze Out (FO) is an important phase of dynamical reactions. The FO is a kinetic process and thus the phase-space (PS) distribution of post FO particles can be obtained from kinetic FO calculations. The connection of the kinetic description of this process and the Boltzmann Transport Equation (BTE) raised considerable attention recently [1,2]. One would think that FO can be handled perfectly by using the BTE, which may describe both equilibrium and non-equilibrium processes in a 4-dimensional space-time (ST) volume element like FO layer. Our aim is to analyse the situation and point out the physical causes, which limit the applicability of the BTE for describing FO and some other processes [3].

FO is usually assumed to happen on sharp 3-dimensional ST hypersurfaces. However, the FO-fronts or FO-layers are not necessarily narrow, but typically they have a characteristic direction or normal,  $d^3\sigma^{\mu}$ , which can be time-like or space-like. It is more realistic to assume a continuous, 4-volume FO in a layer (or domain) of the ST. At the inside boundary of this layer there are only interacting particles, while at the outside boundary hypersurface all particles are frozen out and no interacting particles remain. Assuming that the boundaries of this layer are approximately parallel and neglecting derivatives in all directions other than  $d^3\sigma^{\mu}$ , we can consider FO process as 1-dimensional [3,5].

We can derive the BTE from the conservation of charges in a ST domain,  $\Delta^4 x$ , assuming the standard conditions, see for example [4]:

- (i) only binary collisions are considered,
- (ii) we assume "molecular chaos", i.e. that the number of binary collisions at position x is proportional to  $f(x, p_1) \times f(x, p_2)$ ,
- (iii) f(x,p) is a smoothly varying function compared to the mean free path (m.f.p.).

We have to take into account that particles can scatter into the phase space (PS) volume element around p, or can scatter out from this volume element, described by Gain- and Loss- collision terms in the BTE. In these terms we consider elementary binary collisions where in the initial state two particles collide with momenta  $p_1$  and  $p_2$  into a final state of two particles with momenta  $p_3$  and  $p_4$ . In case of the Gain term the particle described by the BTE, with momentum p (without an index), is one from the two final state particles, while in case of the loss term this particle is one of the initial state particles. This is indicated by the indexes of the invariant transition rate,  $W_{34}^{p_2}$  and  $W_{p2}^{p_3}$  correspondingly [4]. We integrate over the momenta of the other three particles participating in this binary collision. We use the following notation for the PS integrals:

$$D_{123} \equiv \frac{d^3 p_1}{p_1^0} \frac{d^3 p_2}{p_2^0} \frac{d^3 p_3}{p_2^0} \,.$$

So finally we can write BTE in the following way:

$$p^{\mu}\partial_{\mu}f(x,p) = \frac{1}{2} \int D_{234}W_{34}^{p2} \left[ f(x,p_3)f(x,p_4) - f(x,p)f(x,p_2) \right] . \tag{1}$$

Here we want to underline the important symmetry properties of the invariant transition rate, namely  $W_{12}^{34} = W_{21}^{34} = W_{34}^{12}$ . This transition rate can be expressed as  $W_{12}^{34} = s\sigma_{diff}\delta(p_1 + p_2 - p_3 - p_4)$ , where  $\delta$ -function enforces energy-momentum conservation,  $s = (p_1 + p_2)^2$  and  $\sigma_{diff}$  is a differential cross section, here we will assume that it is momentum independent (so the total cross section  $\sigma = 4\pi\sigma_{diff}$ ).

However, the usual structure of the collision terms in the BTE is not adequate for describing rapid process in a layer with a thickness comparable with the m.f.p. If we assume the existence of such a layer this immediately contradicts assumption (iii): the change is not negligible in the direction of  $d^3\sigma^{\mu}$  (normal to the layer). The assumption of "molecular chaos" (ii) is also violated in such a process, because number of collisions is not proportional to  $f(x, p_1) \times f(x, p_2)$ , but it is delocalized in the normal direction with  $f(x_1, p_1) \times f(x_2, p_2)$ , where  $x_k$  is the origin of colliding particles, i.e., the ST point where the colliding particles were colliding last.

Based on the above considerations, one might conclude that the changes of the distribution function are mediated by the transfer of particles, and consequently only slowly propagating changes are possible. If the FO layer propagates slowly, then its normal,  $d^3\sigma^{\mu}$ , must always be space-like. This was a common misconception, where all "superluminous" shock, detonation, deflagration fronts or discontinuities, and FO were considered unphysical based on early studies [6]. However, later it was shown that discontinuous changes may happen simultaneously in spatially neighbouring points, i.e. the normal of the discontinuity-hypersurface can be time-like [7]. This applies to the FO process also. Thus, the direction of characteristic or dominant change,  $d^3\sigma^{\mu}$ , may be both space-like and time-like in the FO process.

From the all processes mentioned above (i.e. shocks, detonations, deflagrations etc.) the FO is the most special one, because the number of interacting particles is constantly decreasing as the FO proceeds, correspondingly the m.f.p. is increasing. In fact, it reaches infinity when the FO is completed. This means that we can not make the FO in finite layer of any thickness smooth enough to be modeled with the BTE! It is also obvious that

if FO has some characteristic length scale, it is not proportional to the m.f.p., because the m.f.p. increases as the density of interacting component becomes smaller, while the FO becomes faster in this limit, so its characteristic scale should decrease.

To describe that the PS distributions which change rapidly, faster than the m.f.p., we suggest a Modified Boltzmann Transport Equation (MBTE):

$$p^{\mu}\partial_{\mu}f(x,p) = \frac{1}{2} \int D_{234} W_{34}^{p2} \left[ \overline{f(x,p_3)}^x \overline{f(x,p_4)}^x - \overline{f(x,p)}^x \overline{f(x,p_2)}^x \right], \tag{2}$$

where  $\overline{f(x, p_i)}^x$  is an average over all possible origins of the particle in the backward lightcone of the ST point  $x = (t, \vec{x})$ :

$$\frac{f(x,p)}{f(x,p)} = \frac{\int_{t_0}^t dt_1 \int d^3x_1 \delta^3(\vec{x} - \vec{x}_1 - \vec{v}(t-t_1)) f(x_1, p) e^{-\int_{t_1}^t dt_2 \int d^3x_2 \sigma n(x_2) v \delta^3(\vec{x}_2 - \vec{x}_1 - \vec{v}(t_2 - t_1))}}{\int_{t_0}^t dt_1 \int d^3x_1 \delta^3(\vec{x} - \vec{x}_1 - \vec{v}(t-t_1)) e^{-\int_{t_1}^t dt_2 \int d^3x_2 \sigma n(x_2) v \delta^3(\vec{x}_2 - \vec{x}_1 - \vec{v}(t_2 - t_1))}}, (3)$$

where  $\delta^3(\vec{x} - \vec{x}_1 - \vec{v}(t - t_1))$  fixes the ST trajectory along which the particles with given momentum can reach the ST point x, time  $t_0$  is given by the initial or boundary conditions,  $\vec{v} = \vec{p}/p^0$  ( $v = |\vec{v}|$ ), and the exponential factor accounts for the probability not to have any other collision from the origin point  $x_1$  till x. In the arguments of exponents n(x) is the particle density in the calculational frame,  $n(x) = \int d^3p f(x,p)$ ,  $\sigma$  is the total scattering cross section. After performing integrations over  $d^3x$  with a help of  $\delta$ -functions we can write MBTE equation in the form:

$$p^{\mu}\partial_{\mu}f = \int \mathcal{D}_{234}^{t_1t_2} \left[ f(t_1, p_3)G(t_1, p_3)f(t_2, p_4)G(t_2, p_4) - f(t_1, p)G(t_1, p)f(t_2, p_2)G(t_2, p_2) \right], (4)$$

where

$$\mathcal{D}_{234}^{t_1 t_2} = \frac{1}{2} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \int D_{234} W_{34}^{p2}, \quad f(t_1, p) = f(t_1, \vec{x} - \vec{v}(t - t_1), p),$$
 (5)

$$G(t_1, p) = \frac{e^{-\int_{t_1}^t dt_2 \sigma n(t_2, \vec{x} - \vec{v}(t - t_2))v}}{C(x, p)}, \quad C(x, p) = \int_{t_0}^t dt_1 e^{-\int_{t_1}^t dt_2 \sigma n(t_2, \vec{x} - \vec{v}(t - t_2))v}.$$
 (6)

The obvious limit in which MBTE is reduced to BTE is a completely homogeneous ST distribution function (i.e. no external forces, no boundaries). Another possibility is a thermodynamical limit,  $\lambda = 1/\sigma n \to 0$ , when the exponential factors (6) will be reduced to  $\sim \delta(t - t_{1,2})$ , reproducing the BTE after  $t_1, t_2$  integrations.

For the BTE the symmetry of the invariant transition rate lead to the consequence that local conservation laws can be derived from the original BTE, i.e.

$$\partial_{\mu}T^{\mu\nu} = 0$$
 and  $\partial_{\mu}N^{\mu} = 0$ 

were T and N are given as momentum-integrals over the single particle PS distribution. Although now we have delocalized the equations, the local conservation laws can be still derived in the same way. Let us create a quantity  $\Psi_k(x) = a(x) + b(x)p_k^{\mu}$ , which is conserved in the binary collisions  $12 \to 34$ , i.e.  $\Psi_1 + \Psi_2 = \Psi_3 + \Psi_4$ . Now let us study the quantity F:

$$F = \int \frac{d^3 p_1}{p_1^0} \Psi_1 p_1^{\mu} \partial_{\mu} f(x, p_1) =$$

$$= \int \mathcal{D}_{1234}^{t_1 t_2} \Psi_1 \left[ f(t_1, p_3) G(t_1, p_3) f(t_2, p_4) G(t_2, p_4) - f(t_1, p) G(t_1, p_1) f(t_2, p_2) G(t_2, p_2) \right] . (7)$$

Using the symmetry of the transition rate it is easy to show [4] that

$$F = \int \mathcal{D}_{1234}^{t_1 t_2} f(t_1, p_3) G(t_1, p_3) f(t_2, p_4) G(t_2, p_4) (\Psi_1(x) + \Psi_2(x) - \Psi_3(x) - \Psi_4(x)) = 0 . (8)$$

Now if we choose  $\Psi_k = q$ , where q is a conserved charge, we obtain the charge conservation, and if we choose  $\Psi_k = p_k^{\nu}$  we obtain the energy-momentum conservation.

The very essential property of the BTE is the Boltzmann H-theorem. In order to study the entropy 4-current,

$$S^{\mu} = \int \frac{d^3 p_1}{p_1^0} p_1^{\mu} f(x, p_1) \left( \log \left( f(x, p_1) \right) - 1 \right) ,$$

we have to choose  $\Psi_k(x) = \log(f(x, p_k))$ , which is not a conserved quantity. Nevertheless, repeating the same steps as for eq. (8) we obtain:

$$S^{\mu}_{,\mu} = \int \mathcal{D}_{1234}^{t_1 t_2} f(t_1, \vec{x} - \vec{v}_3(t - t_1), p_3) G(t_1, p_3) f(t_2, \vec{x} - \vec{v}_4(t - t_2), p_4) G(t_2, p_4) \times \\ \times \log \left( \frac{f(x, p_1) f(x, p_2)}{f(x, p_3) f(x, p_4)} \right) . (9)$$

The usual way of proving that  $S^{\mu}_{,\mu} \geq 0$  does not work for MBTE because of the delocalized integral kernel,  $f(t_1, \vec{x} - \vec{v}_3(t - t_1), p_3)G(t_1, p_3) f(t_2, \vec{x} - \vec{v}_4(t - t_2), p_4)G(t_2, p_4)$ . The behaviour of the entropy current in MBTE is a subject of the future studies. Nevertheless, the condition of the adiabatic expansion  $(S^{\mu}_{,\mu} = 0)$  is the same as for BTE, namely  $f(x, p_1)f(x, p_2) = f(x, p_3)f(x, p_4)$ .

In conclusion, we have shown that the basic assumptions of BTE are not satisfied during the FO process and other very fast processes, and, thus, the description should be modified if used for such a modeling. We suggest the Modified Boltzmann Transport Equation and study some of its properties. Although the use of MBTE makes kinetical FO description much more complicated, some simplifications can be done [3] and the simplified model can be used for the qualitative understanding of the basic FO features.

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